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1977 J. Phys. A: Math. Gen. 10 L111

(http://iopscience.iop.org/0305-4470/10/6/003)

J. Phys. A: Math. Gen., Vol. 10, No. 6, 1977. Printed in Great Britain. © 1977

LETTER TO THE EDITOR

How many figure eights are there? Some bounds

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Received 10 March 1977

Abstract. This letter gives a short proof that the number of figure eights weakly embeddable in a lattice is at least comparable to, and may be much larger than, the number of polygons, when the number of edges in the graph is sufficiently large.

Some years ago we investigated the numbers of figure eights, weakly embeddable in the square lattice, using exact enumeration and Monte Carlo methods (Whittington and Valleau 1970). We suggested, on the basis of these results, that when the graphs have a sufficiently large number of edges the number of figure eights would be at least comparable to, and might be enormously larger than, the number of polygons. The purpose of this letter is to present some rigorous results which support this.

We consider the square lattice for convenience though it is easy to extend the results to other cases. Let $(m)_0$ be the number of (undirected, unrooted) polygons, per site of the square lattice, with m edges which are weakly embeddable in this lattice. Similarly, let $(m, n)_8$ be the number of figure eights with m edges in one circuit and n edges in the other. The total number of figure eights with m edges will be

$$(m)_{\mathrm{E}} = \sum_{n=2}^{m/4} (m-2n, 2n)_8. \tag{1}$$

Since (Whittington and Valleau 1970)

$$(m, n)_8 \geqslant (m)_0(n)_0 \tag{2}$$

we obtain a lower bound by considering a single term in the summation

$$(m)_{\mathrm{E}} \geqslant (m-4, 4)_8 \geqslant (m-4)_0 \tag{3}$$

since $(4)_0 = 1$ on this lattice. To obtain an upper bound we note that (Whittington and Valleau 1970, Hammersley 1961)

$$(n, m)_8 \leqslant nm(n)_0(m)_0 \tag{4}$$

and

$$(n)_0(m)_0 \leqslant (n+m)_0 \tag{5}$$

so that

$$(m)_{\mathrm{E}} \leqslant \sum_{n=2}^{m/4} 2n(m-2n)(2n)_0(m-2n)_0 \leqslant m^3(m)_0. \tag{6}$$

Now the number of polygons $(m)_0$ obeys the limit (Hammersley 1961)

$$\lim_{m\to\infty} m^{-1} \ln (m)_0 = \ln \mu \tag{7}$$

where μ is the effective coordination number of the lattice. The results (3) and (6) show that the same limit is obeyed by the number of figure eights,

$$\lim_{m\to\infty} m^{-1} \ln (m)_{\mathrm{E}} = \ln \mu. \tag{8}$$

There is good numerical evidence (Martin *et al* 1967) that

$$(m)_0 \sim m^{\beta} \mu^{m} \tag{9}$$

and (3), (6) and (9) imply that

$$(m)_{\mathrm{E}} \sim m^{\alpha} \mu^{m} \tag{10}$$

with

$$\beta \leqslant \alpha \leqslant \beta + 3. \tag{11}$$

In two dimensions the best numerical estimate of β would then give

$$-\tfrac{5}{2} \leqslant \alpha \leqslant \tfrac{1}{2}. \tag{12}$$

Thus, for large numbers of edges, the number of figure eights is at least comparable to, and may well be much larger than, the number of polygons.

In fact the upper bound can be improved by comparing the numbers of figure eights and simple chains (Whittington *et al* 1975). If a pair of collinear edges meeting at the articulation point of the figure eight are deleted the resulting graph is a simple chain. Moreover, each figure eight gives a distinct simple chain by this construction so that

$$(m)_{\mathrm{E}} \leqslant (m-2)_{\mathrm{C}}. \tag{13}$$

Since it appears that, in two dimensions (Martin *et al* 1967)

$$(m)_{\mathrm{C}} \sim m^{1/3} \mu^{m} \tag{14}$$

we have

$$-\tfrac{5}{2} \leqslant \alpha \leqslant \tfrac{1}{3} \tag{15}$$

in two dimensions.

The authors enjoy the financial support of the National Research Council of Canada.

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